

Severe right Ore sets and universal localisation

Aidan Schofield

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Abstract

We introduce the notion of a severe right Ore set in the main as a tool to study universal localisations of rings but also to provide a short proof of P. M. Cohn's classification of homomorphisms from a ring to a division ring. We prove that the category of finitely presented modules over a universal localisation is equivalent to a localisation at a severe right Ore set of the category of finitely presented modules over the original ring. This allows us to describe the structure of finitely presented modules over the universal localisation as modules over the original ring.

1 Introduction

The main purpose of this paper is to introduce a type of right Ore set in an additive category with cokernels and to demonstrate the use of this notion in two ways. The first is to provide a short proof of P. M. Cohn's characterisation of epimorphisms from a given ring to division rings; the second is to study universal localisation.

The main theorem we prove about universal localisation is that the category of finitely presented modules over a universal localisation is a right Ore localisation of the category of finitely presented modules over the original ring at the severe right Ore generated by the maps between finitely generated projective modules we wish to invert (see section 2 for the definition of a severe right Ore set). To some extent, this result is a surprise since another approach to the study of universal localisation would be to study the derived category of the universal localisation and here a corresponding result fails to be true; the derived category of the universal localisation can fail to be the right perpendicular category to the maps between finitely generated projective modules considered as objects in the derived category (see [5]).

This allows us to give a module-theoretic description of the finitely presented modules over the universal localisation as modules over the original ring. From this we can give a description of the kernel of the homomorphism from a module to the induced module over the universal localisation. Although this answer is useful, there are many situations where we want a simpler condition. Specifically we should like to be able to say that this kernel is simply the torsion submodule with respect to the torsion theory generated by the cokernels of the

maps between finitely generated projective modules we invert. This is false in general; however, we provide a simple and fairly general sufficient condition on the universal localisation for this to hold. When this does hold we can provide detailed information about the universal localisation so we investigate these particular universal localisations further in the final section. In particular, we show that for these universal localisations $\mathrm{Tor}_i^R(R_\Sigma, R_\Sigma)$ vanishes for $i > 0$ which is the condition required in [5] so that the derived category of the universal localisation should be the right perpendicular category to the maps between finitely generated projective modules considered as objects in the derived category and hence to construct a long exact sequence for universal localisation in algebraic K -theory as demonstrated in [5].

In a subsequent paper we shall use these results to describe the universal localisations of hereditary rings very precisely. We can find all possible universal localisations in terms of suitable subcategories of the category of finitely presented modules over the original ring; we can then describe the category of finitely presented bound modules over the universal localisation as being equivalent to a suitable subcategory of finitely presented bound modules over the original ring and we can describe the finitely generated projective modules over the universal localisation in terms of the submodules of the cokernels of the maps we invert.

2 Severe right Ore sets

We recall the definition of a right Ore set in a small additive category \mathbf{A} . A set of maps σ in a small additive category is said to be a right Ore set if the following conditions are satisfied:

1. The set σ contains all isomorphisms in \mathbf{A} .
2. (*Closed under composition*). If $s, t \in \sigma$ and st exists then $st \in \sigma$.
3. (*Common right multiples exist*). If $s: A \rightarrow B$ lies in σ and $a: A \rightarrow C$ is some map in the category, there exist $t \in \sigma$ and a map b such that $at = sb$.
4. (*σ is right revengeful*). Suppose $s \in \sigma$ and a is a map such that $sa = 0$; then there exists $t \in \sigma$ such that $at = 0$.

We say that two maps a, b in a category are *isomorphic* if there exist isomorphisms u, v such that $b = uav$.

Although it is certainly possible in the definition of a right Ore set to get by without the first assumption and to modify the second assumption to the condition that st is isomorphic to a map in σ , we gain nothing by doing so.

Given a right Ore set in an additive category \mathbf{A} , we are able to describe the maps in the category \mathbf{A}_σ very precisely. Every map may be written in the form as^{-1} and $as^{-1} = bt^{-1}$ if and only if there exist maps $u, v \in \sigma$ such that $au = bv$ and $su = tv$; in particular, $as^{-1} = 0$ if and only if there exists $u \in \sigma$ such that $au = 0$. We leave it to the reader to check that this is true or to consult [3].

We use the notation $[A]$ for the image of A in the category \mathbf{A}_σ where σ is a right Ore set in \mathbf{A} .

Before stating the next theorem, we should point out that we shall be working with additive categories with cokernels as our standard type of category for much of this paper. The reason for this is that the category of finitely presented modules over a ring R is of this type and this category is only an abelian category if the ring is coherent. We shall use the notation $\mathbf{fpmod}(R)$ for this category of finitely presented modules over the ring R . We may and will regard $\mathbf{fpmod}(R)$ as a small category.

Given a functor between additive categories with cokernels we say that this functor is right exact if and only if it preserves cokernels. We shall occasionally say that a sequence $A \rightarrow B \rightarrow C \rightarrow 0$ is exact by which we mean simply that the map from B to C is the cokernel of the map from A to B .

We note the following standard lemma which is usually stated in the context of the full module category over the ring R and a right exact functor to an abelian category but whose proof is identical and obvious in this context.

Lemma 2.1. *Let $\phi: \mathbf{fpmod}(R) \rightarrow \mathbf{A}$ be a right exact functor where \mathbf{A} is an additive category with cokernels. Then ϕ is naturally equivalent to $\otimes_R \phi(R)$.*

Theorem 2.2. *Let \mathbf{A} be an additive category with cokernels and let σ be a right Ore set. Then \mathbf{A}_σ is an additive category with cokernels and the functor from \mathbf{A} to \mathbf{A}_σ is right exact.*

Proof. Let $b: B \rightarrow C$ be the cokernel of $a: A \rightarrow B$. Let $s: D \rightarrow B$ lie in σ . We consider the map sb in \mathbf{A}_σ which we hope is the cokernel of as^{-1} . Let ct^{-1} be a map such that $as^{-1}ct^{-1} = 0$. There exist $c' \in \mathbf{A}$ and $s' \in \sigma$ such that $sc' = cs'$ in \mathbf{A} and so $s^{-1}c = c's'^{-1}$ and $0 = ac'(ts')^{-1}$. Hence there exists $u \in \sigma$ such that $ac'u = 0$ in \mathbf{A} . So $c'u = bd$ for some map d in \mathbf{A} . Therefore $ct^{-1} = sc's'^{-1}t^{-1} = sc'u(ts'u)^{-1} = sbd(ts'u)^{-1}$ as needed. Hence sb is the cokernel of as^{-1} . Thus \mathbf{A}_σ is an additive category with cokernels and the functor from \mathbf{A} to \mathbf{A}_σ is right exact since the cokernel of $a = a1^{-1}$ is still b . \square

Now suppose that \mathbf{A} is an additive category with cokernels. Let $a: A \rightarrow B$ and $b: A \rightarrow C$ be maps in \mathbf{A} . Let $\begin{pmatrix} b' \\ a' \end{pmatrix}: B \oplus C \rightarrow \text{cok}(a \ b)$ be the canonical map to the cokernel; then we call a' the *pushout* of a along b ; similarly, b' is the pushout of b along a . By definition, $ab' = -ba'$.

Let σ be a set of maps in the small category \mathbf{A} . We say that σ is a severe right Ore set if it satisfies the following conditions:

1. The set σ contains all isomorphisms.
2. It is closed under composition.
3. (*Closed under pushout*). Given $s: A \rightarrow B$ in σ and $a: A \rightarrow C$, the pushout of s along a lies in σ .
4. (*Severely right revengeful or closed under cokernels of right killers*). If $s: A \rightarrow B$ is in σ and $a: B \rightarrow C$ is a map such that $sa = 0$ then $t: C \rightarrow \text{cok } a$, the canonical map to the cokernel, also lies in σ .

The last two conditions imply directly the corresponding conditions for a right Ore set so that the name is justified. There are two good reasons for considering this definition. Firstly, because a severe right Ore set is defined by closure conditions, it is possible to talk of the severe right Ore set generated by a set of maps. The second reason is the following lemma.

Lemma 2.3. *Let $\phi: R \rightarrow S$ be a ring homomorphism. Let σ be the set of maps $\{s \in \mathbf{fpmod}(R) : s \otimes_R S \text{ is an isomorphism}\}$. Then σ is a severe right Ore set.*

Proof. It is clearly closed under composition and contains all isomorphisms. Now suppose that $s: A \rightarrow B$ lies in σ . Let $a: A \rightarrow C$ be some map. Let $f = \begin{pmatrix} a' \\ s' \end{pmatrix}: B \oplus C \rightarrow D$ be the cokernel of $(s \ a)$. Then, since ${}_-\otimes_R S$ is right exact, $f \otimes_R S$ is still the cokernel of $(s \ a) \otimes_R S$ and so $s' \otimes_R S$ is the pushout of $s \otimes_R S$ along $a \otimes_R S$. However the pushout of an isomorphism is an isomorphism. So σ is closed under pushout.

Now suppose that $s: A \rightarrow B$ is in σ and $a: B \rightarrow C$ is a map such that $sa = 0$. Let $t: C \rightarrow D$ be the cokernel of a . Since $s \otimes_R S$ is an isomorphism, $a \otimes_R S = 0$. Since ${}_-\otimes_R S$ is right exact, $t \otimes_R S$ is the cokernel of $a \otimes_R S = 0$ and so is an isomorphism. Thus σ is severely right revengeful as well and hence is a severe right Ore set. \square

The reader may check that the proof above actually shows that for any right exact functor between small additive category with cokernels the set of maps inverted by the functor must be a severe right Ore set.

Let Σ be a set of maps between finitely generated projective modules over the ring R . Then we can define σ to be the severe right Ore set generated by Σ in $\mathbf{fpmod}(R)$. We can then form the category $\mathbf{fpmod}(R)_\sigma$. Our main aim in this paper is to show that this category is naturally equivalent to $\mathbf{fpmod}(R_\Sigma)$ via the natural functor induced by $\otimes_R R_\Sigma: \mathbf{fpmod}(R) \rightarrow \mathbf{fpmod}(R_\Sigma)$ which inverts Σ and so by lemma 2.3 also inverts σ . We return to this kind of severe right Ore set later but for the moment we return to the general case.

Let σ be a severe right Ore set in the category $\mathbf{fpmod}(R)$ we define a functor Γ from $\mathbf{fpmod}(R)_\sigma$ to $\mathbf{Mod}(R)$ which we shall refer to as the *realisation functor* by $\Gamma = \text{Hom}_{\mathbf{fpmod}(R)_\sigma}([R], _)$. We shall see that the realisation functor is a full and faithful functor. We begin by describing the image of $[M]$ under Γ . Let ${}_M\sigma$ be the set of maps in σ whose domain is M . We have a directed system of modules $\{M_s\}$ indexed by ${}_M\sigma$ where a morphism from M_s to M_u is given by $t: M_s \rightarrow M_u$ where $t \in \sigma$ and $u = st$. There is an initial object $M = M_1$ since $1_M \in \sigma$. Moreover given $s, t \in {}_M\sigma$, the pushout diagram

$$\begin{array}{ccc} M & \xrightarrow{s} & M_s \\ \downarrow t & & \downarrow s' \\ M_t & \xrightarrow{t'} & M_u \end{array}$$

where $u = st' = s't$ lies in our system because σ is closed under pushout and composition. We define M_σ to be the direct limit of this system $\varinjlim_{s \in {}_M\sigma} M_s$.

Let M and N be finitely presented modules. Because M is finitely presented $\text{Hom}(M, N_\sigma) = \varinjlim_{t \in N\sigma} \text{Hom}(M, N_t)$. We define $\lambda_t: \text{Hom}(M, N_t) \rightarrow \text{Hom}([M], [N])$ by $\lambda_t(f) = ft^{-1}$. Given a map in σ , $u: N_t \rightarrow N_{tu}$, we see that $(M, u)\lambda_{tu} = \lambda_t$ since $(f)(M, u)\lambda_{tu} = fu(tu)^{-1} = ft^{-1}$. Thus we obtain a map $\lambda: \varinjlim_{t \in N\sigma} \text{Hom}(M, N_t) \rightarrow \text{Hom}([M], [N])$ which is visibly surjective.

Lemma 2.4. *Let M and N be finitely presented modules. Then the map $\lambda: \varinjlim_{t \in N\sigma} \text{Hom}(M, N_t) \rightarrow \text{Hom}([M], [N])$ is an isomorphism. So $\text{Hom}(M, N_\sigma) \cong \text{Hom}([M], [N])$.*

Proof. Suppose that $\lambda(k) = 0$. Let $f \in \text{Hom}(M, N_t)$ represent k . So $ft^{-1} = 0$ in $\text{fpmod}(R)_\sigma$. Then there exists $u \in \sigma$ such that $fu = 0$ but then $(f)(M, u) = fu = 0$ and so $k = 0$. Thus λ is injective and hence bijective. \square

Our first use of this is to identify the images of objects under Γ .

Lemma 2.5. *Let $\Gamma: \text{fpmod}(R)_\sigma \rightarrow \text{Mod}(R)$ be the realisation functor. Then $\Gamma([M]) \cong M_\sigma$.*

Proof. By the previous lemma we see that $\Gamma(M) = \text{Hom}([R], [M]) = \text{Hom}(R, M_\sigma) = M_\sigma$. \square

We denote the map from M_s to M_σ by ι_s . In particular ι_1 is the natural map from M to M_σ .

Lemma 2.6. *Let M and N be finitely presented modules. Then the homomorphism $\iota_1: M \rightarrow M_\sigma$ induces an isomorphism $(\iota_1, N_\sigma): \text{Hom}(M_\sigma, N_\sigma) \cong \text{Hom}(M, N_\sigma)$.*

Proof. Let $f: M_\sigma \rightarrow N_\sigma$ be a homomorphism such that $\iota_1 f = 0$. Consider the map $\iota_s f: M_s \rightarrow N_\sigma$. Let $c_s: M_s \rightarrow T_s$ be the cokernel of s . So $\iota_s f = c_s f'$ for some map f' from T_s to N_σ . Then since T_s is finitely presented, we can choose t so that f' factors through N_t ; thus $f' = f_1 \iota_t$. Now $c_s f_1$ is a homomorphism from M_s to N_t such that $sc_s f_1 = 0$ and since σ is a severe right Ore set, the cokernel of $c_s f_1 = 0$, $u: N_t \rightarrow N_{tu}$ lies in σ . So $\iota_t = u \iota_{tu}$. Hence

$$\iota_s f = c_s f' = c_s f_1 \iota_t = c_s f_1 u \iota_{tu} = 0$$

and since this is true for all s we deduce that $f = 0$. So (ι_1, N_σ) is injective.

Now let $\phi: M \rightarrow N_\sigma$ be a homomorphism. Then since M is finitely presented ϕ factors through some N_t ; that is, $\phi = f_1 \iota_t$; for each $s \in M\sigma$, we consider the pushout diagram:

$$\begin{array}{ccc} M & \xrightarrow{s} & M_s \\ f_t \downarrow & & \downarrow f_s \\ N_t & \xrightarrow{s'} & K_s \end{array}$$

Then $s' \in \sigma$ and so $K_s = N_{ts'}$ and we define $\phi_s: M_s \rightarrow N_\sigma$ to be $f_s \iota_{ts'}$. These maps fit together to give a homomorphism from M_σ to N_σ which restricts to ϕ on M which completes the proof. \square

This allows us to conclude that the realisation functor is full and faithful so that $\mathbf{fpmod}(R)_\sigma$ can be thought of as a category of modules or as we shall find useful later, the category of modules may be thought of as a right Ore localisation of the category of finitely presented modules over R .

Theorem 2.7. *The functor $\Gamma: \mathbf{fpmod}(R)_\sigma \rightarrow \mathbf{Mod}(R)$ is full and faithful.*

Proof. We have shown that $\mathrm{Hom}(\Gamma([M]), \Gamma([N])) = \mathrm{Hom}(M_\sigma, N_\sigma) \cong \mathrm{Hom}(M, N_\sigma)$. However, by lemma 2.4, $\mathrm{Hom}(M, N_\sigma) \cong \mathrm{Hom}([M], [N])$ and these isomorphisms are induced by the functor Γ . \square

We shall return to this line of argument in section 4.

3 Epimorphisms to division rings

We break from our main thread at this point to prove P. M. Cohn's characterisation of epimorphisms from a given ring to division rings. Cohn's characterisation was in terms of prime matrix ideals but we shall prove a characterisation in terms of Sylvester rank functions on finitely presented modules. The equivalence of this with prime matrix ideals is shown in [4]. We refer the reader to [1] or to [4] for the definition of a prime matrix ideal.

A Sylvester rank function on an additive category with cokernels satisfies the following properties. It is a function ρ from isomorphism classes of objects in \mathbf{A} to \mathbb{N} which is additive on direct sums and for every exact sequence $A \rightarrow B \rightarrow C \rightarrow 0$, $\rho(C) \leq \rho(B) \leq \rho(A) + \rho(C)$. We note that if we have a right exact functor ϕ from \mathbf{A} to $\mathbf{mod}(D)$ where D is a division ring then we have an associated Sylvester rank function on \mathbf{A} defined by $\rho(A) = \dim_D \phi(A)$.

We extend the Sylvester rank function to a rank function on maps in the category by defining the rank of the map $\phi: A \rightarrow B$ by the formula $\rho(\phi) = \rho(B) - \rho(\mathrm{cok}(\phi))$. We say that ϕ is ρ -full if and only if $\rho(\phi) = \rho(A) = \rho(B)$. Equivalently, ϕ is ρ -full if and only if $\rho(A) = \rho(B)$ and $\rho(\mathrm{cok}(\phi)) = 0$.

Lemma 3.1. *Let ρ be a Sylvester rank function on the additive category with cokernels \mathbf{A} . Then the set of ρ -full maps is a severe right Ore set.*

Proof. Let Π be the set of ρ -full maps.

Suppose that $\alpha: A \rightarrow B$ and $\beta: B \rightarrow C$ are both ρ -full maps. Then $\rho(A) = \rho(B) = \rho(C)$ and we have an exact sequence $\mathrm{cok}(\alpha) \rightarrow \mathrm{cok}(\alpha\beta) \rightarrow \mathrm{cok}(\beta) \rightarrow 0$. Since $\rho(\mathrm{cok}(\alpha)) = 0 = \rho(\mathrm{cok}(\beta))$, it follows that $\rho(\mathrm{cok}(\alpha\beta)) \leq \rho(\mathrm{cok}(\alpha)) + \rho(\mathrm{cok}(\beta)) = 0$ and so $\rho(\mathrm{cok}(\alpha\beta)) = 0$ and $\alpha\beta$ is ρ -full. Thus Π is closed under composition.

Suppose that $\alpha: A \rightarrow B$ is ρ -full and $\beta: A \rightarrow C$ is some map. We form the pushout diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \beta \downarrow & & \downarrow \beta' \\ C & \xrightarrow{\alpha'} & D \end{array}$$

By construction we have an exact sequence $A \rightarrow B \oplus C \rightarrow D \rightarrow 0$ and so $\rho(B) + \rho(C) = \rho(B \oplus C) \leq \rho(A) + \rho(D)$ and since $\rho(A) = \rho(B)$, it follows that $\rho(C) \leq \rho(D)$. On the other hand, the natural map from $\text{cok}(\alpha)$ to $\text{cok}(\alpha')$ is surjective so that $\rho(\text{cok}(\alpha')) = 0$ and so $\rho(D) \leq \rho(\text{cok}(\alpha')) + \rho(C) = \rho(C)$. Thus α' is ρ -full and Π is closed under pushouts.

Now assume that $\alpha: A \rightarrow B$ is ρ -full and $\beta: B \rightarrow C$ is a map such that $\alpha\beta = 0$. Let $\gamma: C \rightarrow D$ be the cokernel of β . We need to show that $\rho(D) = \rho(C)$ which implies that γ is ρ -full since the cokernel of γ is 0. Since $\alpha\beta = 0$, β induces a map $\beta': \text{cok}(\alpha) \rightarrow C$ such that $\text{cok}(\alpha) \rightarrow C \rightarrow D \rightarrow 0$ is an exact sequence and since $\rho(\text{cok}(\alpha)) = 0$, it follows that $\rho(D) = \rho(C)$ as required.

Thus Π is severely right revengeful and since we have checked all the conditions Π is a severe right Ore set. \square

We are now in a position to prove the main theorem of this section. We show that the right Ore localisation of $\mathbf{fpmod}(R)$ at the severe right Ore set constructed in the last theorem must be the category of finite dimensional vector spaces over a division ring when the rank of the ring itself is 1.

Theorem 3.2. *Let R be a ring and let ρ be a Sylvester rank function on the category of finitely presented R modules such that $\rho(R) = 1$. Let Π be the set of ρ -full maps in $\mathbf{fpmod}(R)$. Then $\mathbf{fpmod}(R_\Pi)$ is equivalent to $\mathbf{mod}(D)$ for some division R -ring D such that for any finitely presented module M , $[M] \cong \rho(M)D$.*

Proof. First of all, we show that $\text{End}_{\mathbf{fpmod}(R)_\Pi}([R])$ is a division ring D . Any map from $[R]$ to itself takes the form of $\alpha\beta^{-1}$ for maps $\alpha: R \rightarrow M$ and $\beta: R \rightarrow M$ where β is ρ -full. It follows that $\rho(M) = 1$. So $\rho(\alpha) = 0$ or 1. If $\rho(\alpha) = 1$ then α is invertible and so is $\alpha\beta^{-1}$. If $\rho(\alpha) = 0$ then $\rho(\text{cok}(\alpha)) = 1$ and the cokernel map is ρ -full. It follows that $\alpha = 0$ in $\mathbf{fpmod}(R)_\Pi$ and so $\alpha\beta^{-1} = 0$ in $\mathbf{fpmod}(R)_\Pi$. Thus every endomorphism of $[R]$ in $\mathbf{fpmod}(R)_\Pi$ is either 0 or invertible. So $\text{End}_{\mathbf{fpmod}(R)_\Pi}([R])$ is a division ring D and the natural ring homomorphism from $\text{End}_R(R)$ to $\text{End}_{\mathbf{fpmod}(R)_\Pi}([R])$ makes D an R -ring.

Next we show that if $\rho(M) = m$ then there is a ρ -full map $\alpha: {}^m R \rightarrow M$ by induction on m . If $m = 0$, then the map from 0 to M is ρ -full. Otherwise, suppose that $m > 0$.

To prove this we need to show that if $\alpha: R \rightarrow M$ is a map such that $\rho(\alpha) = 0$ and $\beta: L \rightarrow M$ is some map then $\rho(\beta \oplus \alpha) = \rho(\beta)$. This follows because the commutative square

$$\begin{array}{ccc} M & \longrightarrow & \text{cok}(\alpha) \\ \downarrow & & \downarrow \\ \text{cok}(\beta) & \longrightarrow & \text{cok}(\beta \oplus \alpha) \end{array}$$

is a pushout diagram and the map from M to $\text{cok}(\alpha)$ is ρ -full since it is surjective and $\rho(\alpha) = 0$ implies that $\rho(M) = \rho(\text{cok}(\alpha))$. It follows that the map from $\text{cok}(\beta)$ to $\text{cok}(\beta \oplus \alpha)$ is ρ -full and in particular $\rho(\text{cok}(\beta \oplus \alpha)) = \rho(\text{cok}(\beta))$. We deduce that $\rho(\beta \oplus \alpha) = \rho(\beta)$.

Now suppose that M is generated by the elements $\{m_1, \dots, m_t\}$. Let $\phi_i: R \rightarrow M$ be the map $\phi(r) = m_i r$. If $\rho(\phi_i) = 0$ for each i then induction and the preceding paragraph shows that the surjective map $\oplus_{i=1}^t \phi_i: {}^t R \rightarrow M$ has ρ -rank 0 and consequently $\rho(M) = 0$. The assumption that $\rho(M) > 0$ implies that we may find some map $\phi: R \rightarrow M$ such that $\rho(\phi) > 0$ and hence $\rho(\phi) = 1$. Let $M' = \text{cok}(\phi)$. Then $\rho(M') = m - 1$ and so by induction there exists a ρ -full map $\mu: {}^{m-1} R \rightarrow M'$. We choose some lifting $\mu': {}^{m-1} R \rightarrow M$ such that the induced map to M' is μ . We consider the map $\phi \oplus \mu': {}^m R \rightarrow M$ whose cokernel is by construction $\text{cok}(\mu)$. Since μ is ρ -full, $\rho(\text{cok}(\phi \oplus \mu')) = \rho(\text{cok}(\mu)) = 0$ and thus $\phi \oplus \mu$ is ρ -full.

Thus we have shown that for every finitely presented module M there exists a ρ -full map from $\rho^{(M)} R$ to M . Thus every object of $\mathbf{fpmod}(R)_\Pi$ is isomorphic to ${}^m[R]$ for some integer m . We also know that $\text{End}_{\mathbf{fpmod}(R)_\Pi}([R]) = D$ a division ring and hence $\text{Hom}_{\mathbf{fpmod}(R)_\Pi}([R], \cdot)$ defines an equivalence of categories between $\mathbf{fpmod}(R)_\Pi$ and $\mathbf{mod}(D)$ as we set out to prove.

Moreover the localisation functor is right exact and therefore the functor we now have via composition from $\mathbf{fpmod}(R)$ to $\mathbf{mod}(D)$ is right exact. By lemma 2.1, it follows that it takes the form $\otimes_R D$ and so for every finitely presented module M , $\dim_D(M \otimes_R D) = \rho(M)$. \square

Thus we have proved the hard direction of Cohn's characterisation of epimorphisms from a ring R to division rings. For completeness we state this theorem

Theorem 3.3. *Let R be a ring. Then the epimorphisms from R to division rings are parametrised by the Sylvester rank functions ρ on $\mathbf{fpmod}(R)$ such that $\rho(R) = 1$.*

The parametrisation is given by associating to such an epimorphism from R to D the Sylvester rank function given by $\rho(M) = \dim_D M \otimes D$.

The inverse map from Sylvester rank functions such that $\rho(R) = 1$ to epimorphisms to a division ring is constructed as follows. To a Sylvester rank function ρ , we associate the ring homomorphism $\phi: R \rightarrow D$ where D is the endomorphism ring of $[R]$ in $\mathbf{fpmod}(R)_\sigma$ where σ is the severe right Ore set of ρ -full maps be finitely presented modules over R .

4 Universal localisation via right Ore localisation

At this point we return to our main study. In this section we begin by showing that the category of finitely presented modules over the universal localisation of R at a set of maps between finitely generated projective modules over R may be obtained as the right Ore localisation of $\mathbf{fpmod}(R)$ at the severe right Ore set σ generated by Σ in $\mathbf{fpmod}(R)$. We then use this to investigate the R module structure of modules in $\mathbf{fpmod}(R_\Sigma)$ by a closer examination of the maps in σ .

Let Σ be a set of maps between finitely generated projective modules over the ring R and let σ be the severe right Ore set generated by Σ . Then consider

the functor $_{-}\otimes_R R_\Sigma: \mathbf{fpmod}(R) \rightarrow \mathbf{fpmod}(R_\Sigma)$. Since it inverts all elements of Σ it must invert σ as well and so we obtain a new functor $\Lambda_\Sigma: \mathbf{fpmod}(R)_\sigma \rightarrow \mathbf{fpmod}(R_\Sigma)$. We shall use the notation \downarrow^Σ for the restriction functor from $\mathbf{Mod}(R_\Sigma)$ to $\mathbf{Mod}(R)$.

Theorem 4.1. *Let Σ be a set of maps between finitely generated projective modules over the ring R and let σ be the severe right Ore set generated by Σ . Then the functor $\Lambda_\Sigma: \mathbf{fpmod}(R)_\sigma \rightarrow \mathbf{fpmod}(R_\Sigma)$ is an equivalence of categories.*

Proof. We begin by showing that $\Lambda_\Sigma \downarrow^\Sigma$ is isomorphic to the realisation functor Γ . We need to show that $M_\sigma \cong M \otimes_R R_\Sigma$. For each map u in the directed system whose limit is M_σ , $u \otimes_R R_\Sigma$ is an isomorphism and hence $M_\sigma \otimes_R R_\Sigma \cong M \otimes_R R_\Sigma$. However, M_σ is an R_Σ module, because if $\alpha: P \rightarrow Q$ lies in Σ , then $(\alpha, M_\sigma): \mathbf{Hom}(Q, M_\sigma) \rightarrow \mathbf{Hom}(P, M_\sigma)$ is the map $(\alpha, [M]): \mathbf{Hom}([Q], [M]) \rightarrow \mathbf{Hom}([P], [M])$ in $\mathbf{fpmod}(R)_\sigma$ after applying the isomorphism of lemma 2.4. But α is an isomorphism in $\mathbf{fpmod}(R)_\sigma$ and so $(\alpha, [M])$ is an isomorphism. Thus M_σ is an R_Σ module and since the homomorphism from R to R_Σ is an epimorphism, $M_\sigma \otimes_R R_\Sigma \cong M_\sigma$. At this stage, we see that Λ_Σ is full and faithful and its image lies in $\mathbf{fpmod}(R_\Sigma)$; to be an equivalence we need that every finitely presented module over R_Σ is isomorphic to some $\Lambda_\Sigma([M]) = M_\sigma$. However, R_σ is R_Σ which is a projective object in the image of \downarrow^Σ . Since Γ is full and faithful, $[R]$ is a projective object of $\mathbf{fpmod}(R)_\sigma$, that is, $\Gamma = \mathbf{Hom}([R], _)$ is right exact. Since $\mathbf{fpmod}(R)_\sigma$ is an additive category with cokernels, the image of Γ and hence Λ_Σ is closed under cokernels and therefore every finitely presented module over R_Σ lies in the image of Λ_Σ as required. \square

Whilst the proof of this theorem is still fresh in the mind of the reader we note that we also proved the following corollary.

Theorem 4.2. *Let Σ be a set of maps between finitely generated projective modules over R . Then for any finitely presented module M over R , $M \otimes_R R_\Sigma \cong M_\sigma = \varinjlim_{s \in {}_M\sigma} M_s$ where ${}_M\sigma$ is the set of maps in the severe right Ore set generated by Σ that begin at M .*

For this to be useful we need to understand the maps in σ and the next theorem gives us such a description. First we introduce some relevant ideas.

Let Σ be a set of maps between finitely generated projective modules over R ; then when we invert Σ we also invert other maps between finitely generated projective modules. Thus if we invert α and β , we also invert $\begin{pmatrix} \alpha & h \\ 0 & \beta \end{pmatrix}$ for any map h between the correct projective modules. We say that a set of maps Θ between finitely generated projective modules is *upper triangularly closed* if for all $\alpha, \beta \in \Theta$ and h such that $\begin{pmatrix} \alpha & h \\ 0 & \beta \end{pmatrix}$ is a map between finitely generated projective modules, then this map lies in Θ . The *upper triangular closure* of Σ , $\overline{\Sigma}$, is the smallest upper triangularly closed set of maps containing Σ . Exactly the same considerations apply to define *lower triangularly closed* sets of maps and the *lower triangular closure* $\underline{\Sigma}$ of a set of maps Σ . Finally, we say that a set of maps is *triangularly closed* if it is both upper and lower triangularly closed

and the *triangular closure* of a set of maps Σ is the smallest triangularly closed set of maps $\tilde{\Sigma}$ containing it. It is clear that all maps in the triangular closure of Σ are inverted when we invert Σ and so $R_{\Sigma} \cong R_{\tilde{\Sigma}} \cong R_{\underline{\Sigma}} \cong R_{\overline{\Sigma}}$. It is a simple matter to check that $\tilde{\Sigma} \subset \sigma$ where σ is the severe right Ore set generated by Σ . It follows that the severe right Ore set generated by $\tilde{\Sigma}$ or by either of $\underline{\Sigma}$ or $\overline{\Sigma}$ is just σ .

In the case where all the maps in Σ are injective then so are the maps in the triangular closure of Σ and the maps in the lower triangular closure of Σ give presentations of all modules in the extension closure of the modules S_{Σ} as do the maps in the upper triangular closure.

We say that t is a *good pushout* if there exists a map τ in the lower triangular closure of Σ such that t is a pushout of τ . We say that u is a *good surjection* if there exists a diagram

$$P \xrightarrow{\tau} P' \xrightarrow{a} N \xrightarrow{u} N'$$

where τ is in the lower triangular closure of Σ , $\tau a = 0$ and u is the cokernel of a . Of course, the conditions imply that a good pushout or surjection lies in σ .

In the case where all elements of Σ are injective we shall see later (or the reader can quickly check) that the set of good pushouts is precisely the set of injective maps between finitely presented modules whose cokernel lies in the extension closure of S_{Σ} .

Theorem 4.3. *Let R be a ring and let Σ be a set of maps between finitely generated projective modules over R . Let σ be the severe right Ore set generated by Σ . Then if $s \in \sigma$ there exists a good pushout t and a good surjection u such that $s = tu$.*

Proof. We prove that the set σ' of maps of the form tu , where t is a good pushout and u is a good surjection, is itself a severe right Ore set and since it contains the generators of σ and lies in σ it must be σ .

Clearly the set of good pushouts is closed under pushout. We show firstly that it is closed under composition too. Consider the diagram below where the left hand and right hand square are pushout diagrams and α and β are in the lower triangular closure of Σ .

$$\begin{array}{ccccc} P & \xrightarrow{\alpha} & Q & \xleftarrow{e} & P_1 & \xrightarrow{\beta} & Q_1 \\ a \downarrow & & b \downarrow & \nearrow f & \downarrow c & & \downarrow d \\ L & \xleftarrow{s} & M & \xlongequal{\quad} & M & \xrightarrow{t} & N \end{array}$$

So s and t are good pushouts. Since $\begin{pmatrix} s \\ b \end{pmatrix}$ is surjective and P_1 is projective we can find maps e, f such that $c = eb + fs$. Now consider the map

$$\begin{pmatrix} a & \alpha & 0 \\ f & e & \beta \end{pmatrix} : P \oplus P_1 \rightarrow L \oplus Q \oplus Q_1$$

and suppose that

$$\begin{pmatrix} a & \alpha & 0 \\ f & e & \beta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

from which we see that $ax + \alpha y = 0$ and so $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s \\ b \end{pmatrix} w$ for some map w since $\begin{pmatrix} s \\ b \end{pmatrix}$ is the cokernel of $\begin{pmatrix} a & \alpha \end{pmatrix}$. Hence $cw + \beta z = fsw + ew + \beta z = 0$. Hence $\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} t \\ d \end{pmatrix} v$ for some map v since $\begin{pmatrix} t \\ d \end{pmatrix}$ is the cokernel of $\begin{pmatrix} c & \beta \end{pmatrix}$. Thus

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} st \\ bt \\ d \end{pmatrix} v$$

which shows that

$$\begin{pmatrix} st \\ bt \\ d \end{pmatrix} \text{ is the cokernel of } \begin{pmatrix} a & \alpha & 0 \\ f & e & \beta \end{pmatrix}$$

and hence st is the pushout of $\begin{pmatrix} \alpha & 0 \\ e & \beta \end{pmatrix}$ along a and hence st is a good pushout. Thus the set of good pushouts is closed under composition.

We show that the set of good surjections is closed under pushout. Consider the diagram

$$\begin{array}{ccccccc} P & \xrightarrow{\alpha} & Q & \xrightarrow{a} & L & \xrightarrow{s} & M \\ & & & & \downarrow b & & \downarrow b' \\ & & & & N & \xrightarrow{s'} & K \end{array}$$

where α lies in the lower triangular closure of Σ , $\alpha a = 0$, s is the cokernel of a and the right hand square is a pushout. Then s' is the cokernel of ab and consequently it too is a good surjection.

At this stage, we know that σ' is closed under pushouts. Consider the diagram

$$\begin{array}{ccccc} K & \xrightarrow{s} & L & \xrightarrow{t} & M \\ \downarrow a & & \downarrow b & & \downarrow c \\ A & \xrightarrow{s'} & B & \xrightarrow{t'} & C \end{array}$$

where s is a good pushout and t is a good surjection and the two squares are pushout diagrams. Then the outer rectangle is also a pushout diagram and so the pushout of st is $s't'$ where s' is a good pushout and t' is a good surjection.

Next we show that the set of good surjections is closed under composition.

Consider the diagram

$$\begin{array}{ccccc}
 & P_1 & \xrightarrow{\beta} & Q_1 & \\
 & \downarrow d & & \downarrow c & \searrow b \\
 P & \xrightarrow{\alpha} & Q & \xrightarrow{a} & L \xrightarrow{s} M \\
 & & & & \downarrow t \\
 & & & & N
 \end{array}$$

where α, β lie in the lower triangular closure of Σ , $\alpha a = 0 = \beta b$, s is the cokernel of a and t is the cokernel of b and we need to construct and describe c and d . Since s is surjective we pick c so that $b = cs$. Then $\beta cs = 0$ and since s is the cokernel of a and P_1 is projective there exists a map d such that $da + \beta c = 0$. Hence,

$$\begin{pmatrix} \alpha & 0 \\ d & \beta \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = 0$$

and if $\begin{pmatrix} a \\ c \end{pmatrix} x = 0$ then since s is the cokernel of a , $x = sy$ for some y and then $by = csy = cx = 0$ and so $y = tz$ and $x = stz$ which proves that st is the cokernel of $\begin{pmatrix} a \\ c \end{pmatrix}$ and so the set of good surjections is closed under composition.

Next we show that if s is a good surjection and t is a good pushout such that st exists, then there exist s' and t' where s' is a good surjection, t' is a good pushout and $st = t's'$. Once this is proved it is clear that σ' is closed under composition.

Consider the diagram

$$\begin{array}{ccccc}
 & P & \xrightarrow{\alpha} & Q & \\
 & \swarrow c & \downarrow a & \downarrow b & \\
 K & \xrightarrow{s} & M & \xrightarrow{t} & N
 \end{array}$$

where the square is a pushout diagram, α lies in the lower triangular closure of Σ and s is a good surjection. Then we can choose c so that $a = cs$. So we form the diagram

$$\begin{array}{ccccc}
 P & \xrightarrow{c} & K & \xrightarrow{s} & M \\
 \alpha \downarrow & & \downarrow t_1 & & \downarrow t \\
 Q & \xrightarrow{d} & L & \xrightarrow{s_1} & N
 \end{array}$$

where each square and the outer rectangle are pushout diagrams. But we see that $st = -t_1 s_1$ where t_1 is a good pushout and s_1 is a good surjection since it is a pushout of a good surjection.

As stated above, this proves that σ' is closed under composition since if s_i are good pushouts and t_i are good surjections for $i = 1, 2$ then $s_1 t_1 s_2 t_2 = s_1 s' t' t_2$

where s' is a good pushout and so is $s_1 s'$ and t' is a good surjection and so is $t' t_2$.

It remains to show that σ' is severely right revengeful.

Consider the diagram

$$\begin{array}{ccccccccc} P & \xrightarrow{\alpha} & Q & & & & & & \\ \downarrow a & & \downarrow b & & & & & & \\ A & \xrightarrow{s} & B & \xrightarrow{t} & C & \xrightarrow{d} & D & \xrightarrow{e} & E \end{array}$$

where the square is a pushout diagram, α lies in the lower triangular closure of Σ , t is a good surjection, $std = 0$ and e is the cokernel of d . Assume for the moment that b is surjective. Then since bt is surjective, e is also the cokernel of btd and $abtd = 0$ so that e is a good surjection and lies in σ' . If b is not surjective then replace the left hand square by

$$\begin{array}{ccc} P \oplus P' & \xrightarrow{\begin{pmatrix} \alpha & 0 \\ 0 & I \end{pmatrix}} & Q \oplus P' \\ \downarrow \begin{pmatrix} a \\ p \end{pmatrix} & & \downarrow \begin{pmatrix} b \\ ps \end{pmatrix} \\ L & \xrightarrow{s} & M \end{array}$$

which is still a pushout diagram and now $\begin{pmatrix} b \\ ps \end{pmatrix}$ is surjective. Thus, in all cases, e lies in σ' and σ' is a severe right Ore set which is what we set out to prove. \square

If we do not assume that the maps in Σ , the set of matrices between finitely generated projective modules, are injective then good pushouts are relatively awkward to interpret; however, good surjections are easy to understand. Let T_α be the cokernel of α . Then the cokernel of a map in the lower triangular closure of Σ is simply a module in the extension closure of the set of modules $\{T_\alpha\}$ and every such module occurs as a cokernel. Therefore good surjections are cokernels of maps from such modules. Of course, it is clear that images of such modules in M must lie in the kernel of the map from M to $M \otimes_R R_\Sigma$. Now suppose that all maps in Σ are injective; then good pushouts again are easy to describe. They are simply injective maps whose cokernels lie in the extension closure of the modules T_α .

Theorem 4.4. *Let Σ be a set of injective maps between finitely generated projective modules over R . Let S_Σ be the set of cokernels of elements of Σ . Let \mathbf{E} be the extension closure of S_Σ . Then a map in the category of finitely presented modules over R is a good pushout if and only if it is injective and its cokernel lies in \mathbf{E} . A map is a good surjection if and only if it is surjective and its kernel is a factor of a module in \mathbf{E} .*

Proof. The maps in the lower triangular closure of Σ are injective and their cokernels lie in \mathbf{E} . Therefore, their pushouts have both these properties.

Conversely, suppose that $f: M \rightarrow N$ is an injective map whose cokernel is $T \in \mathbf{E}$. Choose a map $\alpha: P \rightarrow Q$ in the lower triangular closure of Σ whose cokernel is T . Applying $\text{Hom}(_, M)$ to the short exact sequence $0 \rightarrow P \rightarrow Q \rightarrow T \rightarrow 0$, we see that the map from $\text{Hom}(P, M)$ to $\text{Ext}(T, M)$ is surjective and therefore there exists a map $j: P \rightarrow M$ such that the pushout of α along j is f as required.

Now let $s: M \rightarrow N$ be a good surjection. So there exists a map $\alpha: P \rightarrow Q$ in the lower triangular closure of Σ , a map $a: Q \rightarrow M$, such that $\alpha a = 0$ and s is the cokernel of a . Then the map a induces a map $b: \text{cok } \alpha \rightarrow M$ with the same image as a and so s is also the cokernel of b . \square

We are in a position now to use our description of the severe right Ore set generated by a set of maps between finitely generated projective modules over R as the compositions of good pushouts and good surjections to give module-theoretic information about the induction functor $\otimes_R R_\Sigma$. Firstly, we should like to understand the kernel of the homomorphism from a module M to $M \otimes_R R_\Sigma$ and in particular to understand those modules such that $M \otimes_R R_\Sigma = 0$. Secondly, we should like to understand precisely which maps between finitely generated projective modules over R are inverted by the ring homomorphism from R to R_Σ .

5 Induction

In this section we shall assume that the maps in Σ are all injective. The effect of this is that we can and should replace consideration of Σ by the set of modules that are the cokernels of the elements of Σ . For if Σ' is some different set of injective maps between finitely generated projective modules over R having the same set of cokernels (up to isomorphism) as Σ then clearly $R_\Sigma \cong R_{\Sigma'}$. Thus given a set S of finitely presented modules of homological dimension at most 1 where S is the full subcategory of $\text{fpm}\text{od}(R)$ whose objects are isomorphic to modules in S , we define $R_S \cong R_\Sigma$ to be R_Σ where Σ is some set of injective maps between finitely generated projective modules whose set of cokernels is S .

In the discussion of R_S , we need the notion of a torsion theory. See section 5.1 of [2] for a brief summary. Given any set of modules S , we have an associated torsion theory generated by S ; a module F is torsion-free if $\text{Hom}(N, F) = 0$ for all modules $N \in S$ and T is torsion if $\text{Hom}(T, F) = 0$ for every torsion-free module. The torsion theory is usually thought of as a pair \mathcal{T}, \mathcal{F} where \mathcal{T} is the class of torsion modules and \mathcal{F} is the class of torsion-free modules. In the best cases, one would hope that the torsion submodule of M with respect to the torsion theory generated by S would be the kernel of the homomorphism from M to $M \otimes_R R_S$.

If S is a set of finitely presented modules of homological dimension at most 1, then the torsion modules have a useful description.

Lemma 5.1. *Let S be a set of finitely presented modules of homological dimension at most 1. Let \mathcal{T}, \mathcal{F} be the torsion theory generated by S . Let \mathbf{U} be the full*

subcategory of modules that are factor of modules in \mathbf{E} , the extension closure of S . Then every finitely generated module in \mathbf{T} is isomorphic to some module in \mathbf{U} . If T is a module in \mathbf{T} , then every finitely generated submodule of T lies in a larger submodule isomorphic to a module in \mathbf{U} .

Proof. We show first that \mathbf{U} is closed under extensions. For if we have surjections $s_i: E_i \rightarrow U_i$ for $i = 1, 2$ where $E_i \in \mathbf{E}$ and a short exact sequence $0 \rightarrow U_1 \rightarrow U \rightarrow U_2 \rightarrow 0$ then we form the pullback

$$\begin{array}{ccccccc} 0 & \longrightarrow & U_1 & \longrightarrow & X & \longrightarrow & E_2 \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & U_1 & \longrightarrow & U & \longrightarrow & U_2 \longrightarrow 0 \end{array}$$

which gives a surjection from X to U where X is an extension of E_2 on U_1 . Since E_2 has homological dimension at most 1, the map from $\text{Ext}(E_2, E_1)$ to $\text{Ext}(E_2, U_1)$ is surjective and consequently there exists an extension of E_2 on E_1 , E with a surjection onto X and hence onto U . Thus \mathbf{U} is closed under extensions.

Suppose that T is a torsion module. Then every nonzero factor of T has a nonzero map from some module in S . Let T_1 and T_2 be submodules of T such that for every finitely generated submodule N of T_i , there exists a module $U \in \mathbf{U}$ where $N \subset U \subset T_i$, then the same holds for $T_1 \oplus T_2$ and hence for $T_1 + T_2$ since \mathbf{U} is closed under factors. Also a directed union of such submodules is again such a module. So there exists a maximal such submodule, call it T' . Assume that $T' \neq T$. Let $\phi: E \rightarrow T/T'$ be a nonzero map where $E \in S$. Choose a map between finitely generated projective modules $\alpha: P \rightarrow Q$ such that $\text{cok } \alpha = E$. Choose a map $\beta: Q \rightarrow T$ that lifts ϕ . Then the image of $\alpha\beta$ must lie in T' and therefore there exists a module $U \subset T'$ where $U \in \mathbf{U}$ that contains the image of $\alpha\beta$. Let $\gamma: P \rightarrow U$ be the induced map. Consider the submodule $U' = U + \text{im } \beta$. Then the image of U' in T/T' is $\text{im } \phi$. Also $U \cap \text{im } \beta$ contains $\text{im } \alpha\beta$. It follows that U' is a factor of the module U_1 , the pushout of Q along the map γ in the diagram below

$$\begin{array}{ccccccc} 0 & \longrightarrow & P & \longrightarrow & Q & \longrightarrow & E \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & U & \longrightarrow & U_1 & \longrightarrow & E \longrightarrow 0 \end{array}$$

which shows also that $U_1 \in \mathbf{U}$. Therefore, $U' \in \mathbf{U}$. But any module in \mathbf{U} certainly satisfies the condition that every finitely generated submodule lies in a submodule lying in \mathbf{U} ; therefore, because the sum of two such submodules is also a module of this type so is $T' + U'$ which contradicts the maximality of T' . This contradiction implies that $T' = T$ and proves the second part of the theorem. The first follows at once. \square

Let Σ be a set of injective maps between finitely generated projective modules over R and let S be the set of cokernels of elements of Σ ; so $R_S = R_\Sigma$. It is

worth pointing out that if M is torsion-free with respect to the torsion theory generated by S then it does not follow that M embeds in $M \otimes_R R_S$ since there may be a short exact sequence $0 \rightarrow M \rightarrow N \rightarrow T_1 \rightarrow 0$ where T_1 lies in the extension closure of S and a homomorphism from T_2 in the extension closure of S to N whose image intersects M ; this intersection must then lie in the kernel of the homomorphism from M to $M \otimes_R R_S$. There are conditions which make sure that this does not happen and we shall be considering one such later in this paper but first we introduce a related problem.

We should like to be able to describe the complete set of maps between finitely generated projective modules that become invertible under the ring homomorphism from R to R_Σ . This is equivalent to describing the set of finitely presented modules $\{M\}$ of homological dimension at most 1 over R such that $M \otimes_R R_\Sigma = 0 = \text{Tor}_1^R(M, R_\Sigma)$ by considering their presentations. In fact, we have another way to recognise such modules.

Theorem 5.2. *Let R be a ring and let Σ be a set of injective maps between finitely generated projective modules over R . Let M be a finitely presented module of homological dimension at most 1. Then $\text{Hom}(M, _)$ and $\text{Ext}(M, _)$ vanish on R_Σ modules if and only if $M \otimes R_\Sigma = 0 = \text{Tor}_1^R(M, R_\Sigma)$.*

Proof. Of course, $\text{Hom}(M, _)$ vanishes on R_Σ modules if and only if $M \otimes R_\Sigma = 0$. Let $0 \rightarrow P \rightarrow Q \rightarrow M \rightarrow 0$ be some presentation of M as R module where P, Q are finitely generated projective modules. Applying $\otimes R_\Sigma$ gives an exact sequence $P \otimes R_\Sigma \rightarrow Q \otimes R_\Sigma \rightarrow M \otimes R_\Sigma = 0$ so that the first map must be split surjective and its kernel is $\text{Tor}_1^R(M, R_\Sigma)$. On the other hand, applying $\text{Hom}(_, X)$ for some R_Σ module gives the exact sequence $\text{Hom}(Q, X) \rightarrow \text{Hom}(P, X) \rightarrow \text{Ext}(M, X) \rightarrow 0$ and since $\text{Hom}(K, _) = \text{Hom}(K \otimes R_\Sigma, _)$ on R_Σ modules for any R module we see that $\text{Ext}(M, _) \cong \text{Hom}(\text{Tor}_1^R(M, R_\Sigma), _)$ vanishes on R_Σ modules if and only if $\text{Tor}_1^R(M, R_\Sigma) = 0$. \square

Given a set of injective maps between finitely generated projective modules Σ , we shall use the notation $\mathbf{S}(\Sigma)$ for the full subcategory of $\mathbf{fpmod}(R)$ of modules M of homological dimension at most 1 such that R_Σ inverts their presentations or equivalently $M \otimes R_\Sigma = 0 = \text{Tor}_1^R(M, R_\Sigma)$. We call this the category of R_Σ trivial modules. Clearly $R_{\mathbf{S}(\Sigma)} = R_\Sigma$. We want to describe some obvious closure conditions for this category.

The last closure condition of the following lemma is at first sight a little odd; however it will turn out to be useful to us later.

Lemma 5.3. *Let Σ be a set of injective maps between finitely generated projective modules over the ring R . Then $\mathbf{S}(\Sigma)$ is closed under extensions and closed under kernels of surjective maps. It is also closed under cokernels of injective maps whose cokernel has homological dimension 1.*

Finally, if $\phi: A \rightarrow B$ is a map in $\mathbf{S}(\Sigma)$ whose cokernel has homological dimension 1 then $\text{cok } \phi$ lies in $\mathbf{S}(\Sigma)$ and $\text{im } \phi$ and $\text{ker } \phi$ also lie in $\mathbf{S}(\Sigma)$ whenever they are finitely presented.

Proof. Extensions of modules of homological dimension at most 1 and kernels of surjective maps between modules of homological dimension at most 1 must have homological dimension at most 1 and therefore applying $\text{Hom}(_, X)$ for any R_Σ module X to a relevant short exact sequence shows each of the closure conditions in the first paragraph.

From the short exact sequence $0 \rightarrow \text{im } \phi \rightarrow B \rightarrow \text{cok } \phi \rightarrow 0$, it follows that $\text{im } \phi$ has homological dimension at most 1. Applying $\text{Hom}(_, X)$ for X an R_Σ module to this short exact sequence shows that $\text{Ext}(\text{im } \phi, X) = 0$ for every such X and applying $\text{Hom}(_, X)$ to the short exact sequence $0 \rightarrow \ker \phi \rightarrow A \rightarrow \text{im } \phi \rightarrow 0$ shows that $\text{Hom}(\text{im } \phi, X) = 0$ for every such X ; so $\text{im } \phi \in \mathcal{S}(\Sigma)$ whenever $\text{im } \phi$ is finitely presented. Looking at the short exact sequence $0 \rightarrow \text{im } \phi \rightarrow B \rightarrow \text{cok } \phi \rightarrow 0$ again, we see that $\text{cok } \phi$ is finitely presented and satisfies $\text{Hom}(\text{cok } \phi, X) = 0 = \text{Ext}(\text{cok } \phi, X)$ for every R_Σ module X so that $\text{cok } \phi \in \mathcal{S}(\Sigma)$. Finally, we show by reconsidering the short exact sequence $0 \rightarrow \ker \phi \rightarrow A \rightarrow \text{im } \phi \rightarrow 0$ that $\text{Hom}(\ker \phi, X) = 0 = \text{Ext}(\ker \phi, X)$ for every R_Σ module X and $\ker \phi$ has homological dimension at most 1 so if it is finitely presented then it too must lie in $\mathcal{S}(\Sigma)$. \square

We shall say that a full subcategory of $\mathbf{fpmod}(R)$ whose objects are modules of homological dimension at most 1 satisfying the the closure conditions in this lemma a pre-localising subcategory.

We begin with a characterisation for arbitrary universal localisations of the kernel of the natural map from a finitely presented module to the induced module and of when a finitely presented module becomes the zero module under induction.

Theorem 5.4. *Let \mathbf{E} be a subcategory of $\mathbf{fpmod}(R)$ closed under extensions whose objects have homological dimension at most 1. Let M be a finitely presented module. Then $m \in M$ lies in the kernel of the homomorphism from M to $M \otimes R_{\mathbf{E}}$ if and only if there exist a short exact sequence $0 \rightarrow M \rightarrow N \rightarrow E \rightarrow 0$ and a homomorphism $\phi: E' \rightarrow N$ where $E, E' \in \mathbf{E}$ and m lies in the image of ϕ .*

Further, $M \otimes R_{\mathbf{E}} = 0$ if and only if there exist a short exact sequence $0 \rightarrow M \rightarrow N \rightarrow E \rightarrow 0$ and a homomorphism $\phi: E' \rightarrow N$ where $E, E' \in \mathbf{E}$ and M lies in the image of ϕ .

Proof. The functor $\otimes R_{\mathbf{E}}$ is equivalent to the right Ore localisation at the severe right Ore set Π generated by a set of presentations of the modules in \mathbf{E} . So m lies in the kernel of the homomorphism from M to $M \otimes R_{\mathbf{E}}$ if and only if the map $l_m: R \rightarrow M$ given by $l_m(r) = mr$ becomes the zero map over $R_{\mathbf{E}}$ which holds if and only if there exists a map s in Π such that $l_ms = 0$.

However, $s = iu$ where i is a good pushout and u is a good surjection. The result follows at once from our description of good pushouts and good surjections in theorem 4.4 in the previous section.

The second result follows by taking the identity map on M which must become the zero map over $R_{\mathbf{E}}$ if and only if $M \otimes R_{\mathbf{E}} = 0$. \square

We do not expect to have a better theorem than this for arbitrary universal localisations; however, there is a relatively common situation where we can prove a better theorem.

Theorem 5.5. *Let \mathbf{E} be a pre-localising subcategory of $\mathbf{fpmod}(R)$. Assume that the kernel of any map in \mathbf{E} is a torsion module with respect to the torsion theory generated by \mathbf{E} . Then the kernel of the homomorphism from a finitely presented module M to $M \otimes R_{\mathbf{E}}$ is the torsion submodule of M with respect to the torsion theory generated by \mathbf{E} .*

In particular, if M is torsion-free with respect to this torsion theory then M is an R -submodule of $M \otimes R_{\mathbf{E}}$.

Proof. Using the last theorem, we see that $m \in M$ lies in the kernel of the homomorphism from M to $M \otimes R_{\mathbf{E}}$ if and only if there exist a short exact sequence $0 \rightarrow M \rightarrow N \rightarrow E \rightarrow 0$ and a map $\phi: E' \rightarrow N$ where $E, E' \in \mathbf{E}$ such that the image of ϕ contains m . We consider the induced map from E' to E . By assumption, the kernel of this map K , is torsion with respect to the torsion theory generated by \mathbf{E} and the image of K in N lies in M , is torsion and must contain m . The result follows. \square

There are a couple of other ways in which the condition that the kernel of a map in a pre-localising category should be torsion with respect to the torsion theory generated by the pre-localising category is decisive for us. The next two theorems show that this condition forces the category of $R_{\mathbf{S}}$ -trivial modules to be just \mathbf{S} . Then we show that the ring homomorphism from R to $R_{\mathbf{S}}$ is stably flat which is the condition for the derived category of $R_{\mathbf{S}}$ modules to be the derived category of R modules which in turn implies a localisation sequence in algebraic K -theory.

Theorem 5.6. *Let \mathbf{S} be a pre-localising subcategory of $\mathbf{fpmod}(R)$. Assume that the kernel of any map in \mathbf{E} is a torsion module with respect to the torsion theory generated by \mathbf{E} . Let Σ be a set of presentations of all the modules in \mathbf{S} . Then $\mathbf{S} = \mathbf{S}(\Sigma)$.*

Proof. Let $M \in \mathbf{S}(\Sigma)$. Then by the previous theorem, M is a torsion module with respect to the torsion theory generated by \mathbf{S} . Since \mathbf{S} consists of modules whose homological dimension is at most 1, there exists a short exact sequence $0 \rightarrow N \rightarrow T \rightarrow M \rightarrow 0$ where $T \in \mathbf{S}$. Since $M \in \mathbf{S}(\Sigma)$, $N \otimes R_{\Sigma} \cong \mathrm{Tor}_1^R(M, R_{\Sigma}) = 0$ and so N is also a torsion module with respect to the torsion theory generated by \mathbf{S} and so there exists a surjective map from some $T' \in \mathbf{S}$ to N which gives us a map from T' to T with cokernel M which is a module of homological dimension at most 1. By the definition of a pre-localising subcategory of $\mathbf{fpmod}(R)$, $M \in \mathbf{S}$. \square

We recall that Ranicki and Neeman introduced the notion of a stably flat ring extension. They say that S is a stably flat R ring if and only if $\mathrm{Tor}_i^R(S, S) = 0$ for all i . They show that there is a long exact sequence in algebraic K -theory for universal localisation when R_{Σ} is a stably flat R ring. We note that our

condition on a pre-localising category implies that the universal localisation is often stably flat.

Theorem 5.7. *Let \mathbf{E} be a pre-localising subcategory of $\mathbf{fpmod}(R)$ and assume that kernel of maps in \mathbf{E} are torsion. Then $R_{\mathbf{E}}$ is a stably flat R ring.*

Proof. Let τ be the set of good pushouts whose domain is R . Given $t \in \tau$, we call the codomain of t , M_t . Note that τ is a directed system and we let $\hat{R} = \varinjlim_{t \in \tau} M_t$. Then the map from \hat{R} to $R_{\mathbf{E}} = \hat{R} \otimes R_{\mathbf{E}}$ is surjective and its kernel K is torsion with respect to the torsion theory generated by \mathbf{E} .

It is clear that $\text{Tor}_i^R(\hat{R}, R_{\mathbf{E}}) = 0$ since \hat{R} is a direct limit of the modules M_t and each of these is an extension of R by a module in \mathbf{E} and $\text{Tor}_i^R(E, R_{\mathbf{E}}) = 0$ for every $E \in \mathbf{E}$. Thus it is enough to show that $\text{Tor}_i^R(K, R_{\mathbf{E}}) = 0$. Regarding K as the direct limit of its finitely generated torsion submodules, it is enough to show that $\text{Tor}_i^R(\cdot, R_{\mathbf{E}})$ vanishes on finitely generated modules torsion with respect to the torsion theory generated by \mathbf{E} that are submodules of the modules M_t . We begin by showing that $\text{Tor}_i^R(\cdot, R_{\mathbf{E}})$ vanishes on finitely generated modules torsion with respect to the torsion theory generated by \mathbf{E} that are submodules of the modules in \mathbf{E} .

Let $T \subset F$ be such a module (where $F \in \mathbf{E}$) and choose some short exact sequence $0 \rightarrow L \rightarrow E \rightarrow T \rightarrow 0$ where $E \in \mathbf{E}$. L must be torsion because it is in the kernel of the map from E to F . Firstly, $\text{Tor}_1^R(T, R_{\mathbf{E}}) = L \otimes R_{\mathbf{E}} = 0$ and so $\text{Tor}_1^R(\cdot, R_{\mathbf{E}})$ vanishes on arbitrary torsion submodules of modules in \mathbf{E} . Secondly, $\text{Tor}_{i+1}^R(T, R_{\mathbf{E}}) \cong \text{Tor}_i^R(L, R_{\mathbf{E}})$ and so the assumption that $\text{Tor}_i^R(\cdot, R_{\mathbf{E}})$ vanishes on arbitrary torsion submodules of modules in \mathbf{E} implies that $\text{Tor}_{i+1}^R(\cdot, R_{\mathbf{E}})$ also vanishes on all finitely generated torsion submodules of modules in \mathbf{E} and hence vanishes on arbitrary torsion submodules of modules in \mathbf{E} . Thus we are done by induction.

Now let U be a finitely generated torsion submodule of some M_t . We choose some short exact sequence $0 \rightarrow L \rightarrow E \rightarrow U \rightarrow 0$ where $E \in \mathbf{E}$. Consider the short exact sequence $0 \rightarrow R \rightarrow M_t \rightarrow E_t \rightarrow 0$ where $E_t \in \mathbf{E}$. The kernel of the induced map from E to E_t is a torsion module $K \supset L$ and $K/L \subset R$ from which it follows that $K = L$ and U is a submodule of E_t and we have shown in the previous paragraph that $\text{Tor}_i^R(U, R_{\mathbf{E}}) = 0$. It follows that $\text{Tor}_i^R(K, R_{\mathbf{E}}) = 0$ and hence from the short exact sequence $0 \rightarrow K \rightarrow \hat{R} \rightarrow R_{\mathbf{E}} \rightarrow 0$ that $\text{Tor}_i^R(R_{\mathbf{E}}, R_{\mathbf{E}}) = 0$ for all $i > 0$. \square

Finally we state what our localisation sequence in algebraic K -theory is.

Theorem 5.8. *Let \mathbf{E} be a pre-localising subcategory of $\mathbf{fpmod}(R)$ and assume that kernels of maps in \mathbf{E} are torsion. Then there is a long exact sequence in algebraic K -theory*

$$\dots K_i(\mathbf{E}) \rightarrow K_i(R) \rightarrow K_i(R_{\mathbf{E}}) \rightarrow \dots K_0(R) \rightarrow K_0(R_{\mathbf{E}})$$

Proof. This follows at once from the previous two theorems and [5]. \square

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